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STABILITY OF CERTAIN HOLOMORPHIC MAPS

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A holomorphic map $f: X \to Y$ is said to be *source-stable* (resp. *target-stable*) if any small deformation of X (resp. Y) lifts to a deformation of the triple (f, X, Y). The purpose of this paper is to prove a number of assorted results on the source or target-stability of some particular classes of maps. The main new result (Theorem 3.2) asserts the target stability of a small resolution $f: X \to Y$ of a 3-fold rational singularity, such that the exceptional locus of f is smooth and the canonical bundle K_X is f-ample. This result is related to some recent work of Kollàr and Mori [4] concerning stability of flips. The proof involves a fairly detailed study of the scheme-theoretic exceptional fibre and a vanishing theorem for twisted Kähler differentials on it.

In addition, we will reprove a number of results essentially given in [5] but under additional and unnecessary hypotheses. These include target-stability of "nice" embeddings (Theorem 1.1), source-stability for surjections with vanishing R^1 (Theorem 2.1) and target-stability for maps étale in codimension 2 (Theorem 2.3).

As a general reference on deformation theory of maps, we will use [5]. After [5] was written, the author became aware of the impressive tome [1] by Bingener and Kosarew. While the formalism of [5] should in principle be a special case of that of [1], it is not immediately obvious how to affect the "specialization" in question and it is also possible that, due to our relatively simple context (compact, reduced spaces), some essential simplification has occurred.

0. Preliminaries

In this paper all complex spaces will be assumed compact and reduced, unless otherwise specified. We begin by recalling some formalism from [5]. For a space X, we put $T_X^i = \mathrm{Ext}_X^i(\Omega_X, \mathscr{O}_X)$.

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In [5], we define groups

$$T_f^i = \operatorname{Ext}^i(\delta_1, \delta_0), \qquad i = 0, 1, \cdots,$$

where $\delta_0 : f^*\mathscr{O}_Y \to \mathscr{O}_X$ and $\delta_1 : f^*\Omega_Y \to \Omega_X$ are the natural maps. The group T_f^1 classifies data of the form

$$0 \to \mathscr{O}_X \to A \to \Omega_X \to 0, \quad 0 \to \mathscr{O}_Y \to B \to \Omega_Y \to 0 \qquad \text{(exact)},$$

$$\begin{array}{ccccc} f^*\mathscr{O}_Y & \longrightarrow & f^*B & \longrightarrow & f^*\Omega_Y \\ {\scriptstyle\delta_0}\downarrow & & \downarrow & & \downarrow{\scriptstyle\delta_1} \\ \mathscr{O}_X & \longrightarrow & A & \longrightarrow & \Omega_X \end{array} \qquad \text{(commutative)},$$

and is in 1-1 correspondence with 1st order deformation of f. The group T_f^2 is an obstruction group for deformations of f.

We have the following long exact sequence:

$$(0.1) \qquad \begin{aligned} &0 \to T_f^0 \to T_X^0 \oplus T_Y^0 \to \operatorname{Ext}^0)f(\Omega_Y, \mathscr{O}_X) \\ &\to T_f^1 \to T_X^1 \oplus T_Y^1 \to \operatorname{Ext}_f^1(\Omega_{Y_i}\mathscr{O}_X) \to T_f^2 \to \cdots, \end{aligned}$$

where $\operatorname{Ext}_f^i(\cdot, \cdot)$ are the relative Ext-groups (derived functors of $\operatorname{Hom}_X(f^*\cdot, \cdot))$.

The following simple stability criteria are essentially well known in similar contexts, and will be used constantly in the sequel.

Source-Stability Criterion 0.1. *Let $f : X \to Y$ be a morphism such that the natural map $\alpha_i : T_f^i \to T_X^i$ is surjective for $i = 1$ and injective for $i = 2$. Then f is source-stable.*

Target-Stability Criterion 0.2. *Let $f : X \to Y$ be a morphism such that the natural map $\beta_i : T_f^i \to T_Y^i$ is surjective for $i = 1$ and injective for $i = 2$. Then f is target-stable.*

The point here is the following. Given an nth order deformation ε of X, say, which lifts to a deformation $\tilde{\varepsilon}$ of f, extendability of ε to an $(n+1)$ st order deformation of X is measured by the vanishing of an obstruction in T_X^2; if this obstruction vanishes then by injectivity of α_2 so does the obstruction associated to $\tilde{\varepsilon}$, so that $\tilde{\varepsilon}$ extends to an $(n+1)$ st order deformation as well. Combining this with the fact that all 1st order deformations lift, we conclude that *all* infinitesimal deformations of X lift to deformations of f, hence f is source-stable. The case of target-stability is identical.

1. Embeddings

In this section we will give a target-stability result for embeddings $f: X \hookrightarrow Y$. As the obstruction group for embedded deformations of X within Y is $\operatorname{Ext}^1_X(I/I^2, \mathscr{O}_X)$, $I = \mathscr{I}_{X,Y}$, it is natural to expect that the vanishing of this Ext group is a sufficient condition for the target-stability of f. This, essentially, is what we will prove.

Theorem 1.1. *Let $f: X \hookrightarrow Y$ be an embedding with ideal sheaf $I = \mathscr{I}_{X,Y}$. Suppose no component of X is contained in the singular locus of Y, and that*

$$\operatorname{Ext}^1_X(I/I^2, \mathscr{O}_X) = 0 = T^2_X.$$

Then f is target-stable.

Remark 1.2. If f is a regular embedding, with normal bundle N, then $\operatorname{Ext}^1_X(I/I^2, \mathscr{O}_X) = H^1(N)$. Theorem 1.1 was stated in [5] under some additional, and unnecessary, hypotheses.

Proof of Theorem 1.1. Consider the following usual exact sequence:

$$T^1_f \to T^1_X \oplus T^1_Y \to \operatorname{Ext}^1_f(\Omega_Y, \mathscr{O}_X) \to T^2_f \to T^2_Y \to \cdots$$

$$\beta_1 \qquad\qquad \gamma \qquad\qquad\qquad\qquad \beta_1$$

If we can show γ is surjective, then it follows that β_1 is surjective and β_2 is injective, hence f is target-stable by Criterion 0.2.

To prove surjectivity of γ, note first that

$$\operatorname{Ext}^1_f(\Omega_Y, \mathscr{O}_X) \cong \operatorname{Ext}^1_X(f^*\Omega_Y, \mathscr{O}_X);$$

indeed we have as usual a spectral sequence abutting to the LHS where the only contribution other than the RHS is, by our assumption that no component of X is contained in $\operatorname{sing}(Y)$, of the form $\operatorname{Hom}_X(\text{torsion}, \mathscr{O}_X)$, hence vanishes. For a similar reason, the kernel τ of the natural map

$$I/I^2 \to f^*\Omega_Y$$

must be torsion, and hence $\operatorname{Ext}^1_X(I/(I^2\tau), \mathscr{O}_X)$ injects into $\operatorname{Ext}^1_X(I/I^2, \mathscr{O}_X) = 0$, hence vanishes. Now, by dualizing the exact sequence

$$0 \to (I/I^2)/\tau \to f^*\Omega_Y \to \Omega_X \to 0, \tag{1.1}$$

We conclude the surjectivity of

$$\operatorname{Ext}^1_X(\Omega_X, \mathscr{O}_X) \to \operatorname{Ext}^1_X(f^*\Omega_Y, \mathscr{O}_X),$$

hence of γ.

Remark 1.3. When f fails to be regular, the above Ext group is quite often nonzero (e.g. it *always* is if $\dim X = 1$). Still, the foregoing argument does yield some information in that case. Let γ' denote the natural map $T^1_Y \to \operatorname{Ext}^1_f(\Omega_Y, \mathcal{O}_X)$. Then, as above, a 1st order deformation ε of Y lifts to a deformation of f iff $\gamma'(\varepsilon) \in \operatorname{im}\gamma$, and by the exact sequence (1.1) this holds iff the image of ε in $\operatorname{Ext}^1_X(I/I^2, \mathcal{O}_X)$ vanishes. Making the natural assumption that $H^1((I/I^2)^*) = 0$, where $*$ denotes dual, so that

$$\operatorname{Ext}^1_X(I/I^2, \mathcal{O}_X) \hookrightarrow H^0(\mathcal{E}xt^1_X(I/I^2, \mathcal{O}_X)),$$

we may conclude that ε lifts to a 1st order deformation of f iff ε is locally trivial along the *irregular locus* of f (e.g. when X is smooth this locus is just $X \cap \operatorname{sing}(Y)$). An amusing case to work out is that of a smooth curve X on a singular surface Y, e.g. a surface with generic singularities in $\mathbf{P}^3$.

2. Surjections

From now on, we will consider deformations of surjections $f: X \to Y$. For those, source-stability is generally a very easy matter, as the following, essentially well-known, result shows.

Theorem 2.1. *Let $f: X \to Y$ be a morphism with*

$$f_*\mathcal{O}_X = \mathcal{O}_Y, \qquad R^1 f_*\mathcal{O}_X = 0.$$

Then f is source-stable.

Remark 2.2. This result was given in [5] under the additional, and unnecessary, hypothesis that $R^2 f_*\mathcal{O}_X = 0$.

Proof of Theorem 2.1. Consider the usual exact sequence:

$$T^1_f \to T^1_X \oplus \underbrace{T^1_Y \to \operatorname{Ext}^1_f}_{\gamma_1}(\Omega_Y, \mathcal{O}_X)$$

$$\to T^2_f \to T^2_X \oplus \underbrace{T^2_Y \to \operatorname{Ext}^2_f}_{\gamma_2}(\Omega_Y, \mathcal{O}_X) \cdots.$$

By Criterion 0.1 it will suffice to prove that γ_1 is surjective and γ_2 is injective. But these maps figure in a Leray spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_Y(\Omega_Y, R^q f_*\mathcal{O}_X) \Rightarrow \operatorname{Ext}^i_f(\Omega_Y, \mathcal{O}_X).$$

For $i = 1$, the vanishing of $R^1 f_*\mathcal{O}_X$ yields that γ_1 is, in fact, an isomorphism. For $i = 2$, we get that γ_2 is automatically injective, being an edge-homomorphism. q.e.d.

We turn next to the more subtle question of target-stability of surjections. We begin with the relatively easy case of maps étale in codimension 2.

Theorem 2.3. *Let $f: X \to Y$ be a surjective morphism étale in codimension 2 and flat in codimension 1, and assume moreover that X is locally S_3. Then f is target-stable.*

Remark 2.4. For f finite, this result was first proven by Kollár in his thesis [3], and applied by him to the case of the "canonical index-1 cover" of a variety.

Proof of Theorem 2.3. Arguing as usual, it will suffice to prove that

$$\gamma_i: T^i_X \to \mathrm{Ext}^i_Y(\Omega_Y, \mathcal{O}_X)$$

is surjective for $i = 1$ and injective for $i = 2$. Note first that because f is flat in codimension 1, we have

$$\mathrm{Ext}^i_X(f^*\Omega_Y, \mathcal{O}_X) \simeq \mathrm{Ext}^i_f(\Omega_Y, \mathcal{O}_X), \qquad i \leq 2;$$

indeed in the spectral sequence computing the RHS, the only terms contributing to it other than the LHS are of the form $\mathrm{Ext}^j_X((\mathrm{tor}), \mathcal{O}_X)$ with $j \leq 1$ and (tor) supported in codimension 2, and by an easy result in homological algebra such groups must vanish (X being S_2 would suffice for this).

Now we have an exact sequence

$$0 \to K \to f^*\Omega_Y \xrightarrow{df} \Omega_X \to \Omega_{X/Y} \to 0, \qquad K := \ker(df),$$

and the fact that f is étale in codimension 2 implies that K and $\Omega_{X/Y}$ are both supported in codimension 3, hence

$$\mathrm{Ext}^i_X(K, \mathcal{O}_X) = \mathrm{Ext}^i_X(\Omega_{X/Y}, \mathcal{O}_X) = 0, \qquad i \leq 2.$$

By an easy diagram chase, this implies that the natural map

$$\mathrm{Ext}^i(\Omega_X, \mathcal{O}_X) \to \mathrm{Ext}^i(f^*\Omega_Y, \mathcal{O}_X)$$

is an isomorphism for $i \leq 1$ and injective for $i = 2$, hence so is γ_i.

Remark 2.5. Theorem 2.3 yields, in particular, the target-stability of any birational morphism $f: X \to Y$ when X is S_3 and the exceptional locus of f in X has codimension ≥ 3. It leaves open, however, the subtle and interesting case of birational morphisms having codimension-2 exceptional locus. In this case one clearly cannot expect anything as general as Theorem 2.3. A special case pertaining to 3-folds will be considered in the next section.

3. Canonically positive contractions of 3-folds

The purpose of this section is to prove a target-stability result for small resolutions of 3-folds which resemble numerically the "+-side" of a flip. We begin with a definition.

Definition 3.1. A *canonically positive contraction* of 3-folds is a proper birational morphism $f: X \to Y$ such that X is $\mathbb{Q}$-factorical, Y is normal, $R^1 f_* \mathcal{O}_X = 0$, and f is an isomorphism off a curve $C \subset X$ such that for every irreducible component C_i of C we have $C_i \cdot K_X > 0$.

A class of examples is given by the +-sides of 3-dimensional flips (cf. [2]).

Theorem 3.2. *A canonically positive contraction with smooth exceptional locus, whose source is smooth in a neighborhood of the exceptional locus, is target-stable.*

Remarks 3.3. (i) For $f: X \to Y$ the +-side of a flip, with X being allowed terminal singularities along the exceptional locus, Kollar and Mori [4] have recently proven that f is "weakly target-stable", i.e. that any deformation of Y lifts *after some base-change* to a deformation of f. This raises the intriguing question whether the natural common generalization of [4] and Theorem 3.2 is true: i.e. is any canonically positive contraction with terminal source weakly target-stable?

(ii) As is well known, the theorem is false, even in the weak sense, for "canonically trivial" or "canonically negative" contractions. (But see Remark 3.4.)

(iii) On the other hand, it is reasonable to believe that the theorem is true, and with a similar proof, without the assumption that the exceptional locus C is smooth (in general C will be a "rational forest", i.e. a disjoint union of rational trees).

Proof of Theorem 3.2. Arguing as in the proof of Theorem 2.3, we see that it suffices to prove

$$\mathrm{Ext}^2_X(\Omega_{X/Y}, \mathcal{O}_X) = 0.$$

By Serre duality, this is equivalent to proving

$$H^1(\Omega_{X/Y} \otimes K_X) = 0.$$

Note that $\Omega_{X/Y}$ is a sheaf supported on the exceptional curve C, and there is therefore no loss of generality, first in assuming C is irreducible, and second in replacing X by a small neighborhood of C so that we have

$$H^1(\mathscr{O}_X) = 0, \tag{3.1}$$

$$H^i(\mathscr{F}) = 0, \qquad i > 1, \tag{3.2}$$

for any coherent $\mathscr{O}_X$-module $\mathscr{F}$. Note that it follows from (3.1) and (3.2) that $H^1(\mathscr{O}_C) = 0$, so that $C \simeq \mathbf{P}^1$.

Now put $0 = f(C) \in Y$, $I = \mathscr{I}_{C,X}$, and

$$J = f^*\mathbf{m}_{0,Y} = \operatorname{im}(H^0(I) \otimes \mathscr{O}_X \to \mathscr{O}_X).$$

Thus J is the ideal of $\mathscr{O}_X$ generated by the global sections of I, so that $J \subseteq I$ (the inclusion will be strict unless I/I^2 is seminegative). Finally, put

$$C = \operatorname{Spec}(\mathscr{O}_X/J).$$

Now to begin with, we will identify the relative Kähler differentials $\Omega_{X/Y}$ with the differentials on $\tilde{C}$, i.e. we claim that $\Omega_{X/Y} \simeq \Omega_{\tilde{C}}$. To see this recall that the sheaf $\Omega_{\tilde{C}}$ is characterized by the existence of the universal derivation $d:\mathscr{O}_{\tilde{C}} \to \Omega_{\tilde{C}}$. Now note that the usual derivation $d:\mathscr{O}_X \to \Omega_X$ takes $J = f^*\mathbf{m}_{0,Y}$ to $f^*\Omega_Y$, hence factors through a derivation $\overline{d}:\mathscr{O}_{\tilde{C}} \to \Omega_{X/Y}$, hence by universality a linear map

$$\varphi: \Omega_{\tilde{C}} \to \Omega_{X/Y}$$

such that $\overline{d} = \varphi \circ d$. To go the other way, note that the natural pullback map $\Omega_X \to \Omega_{\tilde{C}}$ vanishes on $f^*\Omega_Y$, hence factors through a map

$$\psi: \Omega_{X/Y} \to \Omega_{\tilde{C}}.$$

By construction, $\psi\circ\varphi$ and $\varphi\circ\psi$ are both the identity on exact differentials dg, hence are the identity, and φ and ψ are inverse isomorphisms.

Now the natural approach to studying the scheme $\tilde{C}$ and proving the required vanishing of $H^1(\Omega_{\tilde{C}} \otimes K_X)$ is to consider a suitable filtration of $\tilde{C}$ given by the thickenings of C of increasing thickness. A strong interpretation of "suitable" is given by the following.

Definition. Let $\tilde{C}$ be a scheme structure on $C = \mathbf{P}^1$. A filtration

$$C = C_1 \subseteq C_2 \subseteq \cdots \subseteq C_r = \tilde{C}$$

of $\tilde{C}$ by subschemes supported on C is said to be *good* if the following condition for ideal sheaves holds:

$$\mathscr{I}_{C_{m-1}, C_m} \simeq l_m \mathscr{O}_C(-1), \qquad m = 2, \cdots, r, \; l_m \geq 0.$$

Now the proof will be concluded by establishing the following two statements.

Claim 1. $\tilde{C}$ as above admits a good filtration.

Claim 2. If $\tilde{C}$ is any scheme structure on $C = \mathbf{P}^1$ admitting a good filtration, and K is a line bundle on $\tilde{C}$ such that $\deg(K|_C) > 0$, then $H^1(\Omega_{\tilde{C}} \otimes K) = 0$.

Proof of Claim 1. We proceed to define a good filtration of $\tilde{C}$. Let $\overline{J}_m \subset \mathscr{O}/I^m$ be the ideal generated by $H^0(I/I^m)$, let $J_m \subseteq \mathscr{O}_X$ be the ideal containing I^m such that $J_m/I^m = \overline{J}_m$, and put $C_m = \operatorname{Spec}(\mathscr{O}_X/J_m)$.

Note that for sufficiently large m the Holomorphic Function Theorem yields $H^0(I/I^m) \simeq H^0(I)$, hence $J_m = J$ and $C_m = \tilde{C}$ eventually. To show $C_\bullet$ is good, it will suffice to prove that

$$J_{m-1}/J_m \simeq l_m \mathscr{O}_C(-1), \qquad m = 2, \cdots.$$

We claim first that

$$H^0(I/J_m) = 0. \tag{3.3}$$

To see this, consider the exact sequence

$$0 \to \overline{J}_m \to I/I^m \to I/J_m \to 0.$$

By construction, $\overline{J}_m$ is a quotient of some copies of $\mathscr{O}_X/I^m$, hence of some copies of $\mathscr{O}_X$, hence by (3.1) and (3.2) we have $H^1(\overline{J}_m) = 0$, so that the map $H^0(I/I^m) \to H^0(I/J_m)$ is surjective; but this map is zero by construction. This proves (3.3). Note that (3.3) implies

$$H^0(J_{m-1}/J_m) = 0.$$

We claim next that J_{m-1}/J_m is actually an $\mathscr{O}_C$-module, i.e. is annihilated by I. To see this consider as before a surjection

$$k(\mathscr{O}_X/I^{m-1}) \to \overline{J}_{m-1} = J_{m-1}/I^{m-1},$$

yielding surjections

$$k\mathscr{O}_X \oplus I^{m-1} \to J_{m-1},$$

$$k(\mathscr{O}_X/J_m) \oplus (I^{m-1}/J_m) \to J_{m-1}/J_m.$$

But as $H^0(J_{m-1}/J_m) = 0$, the induced map $k(\mathscr{O}_X/J_m) \to J_{m-1}/J_m$ must vanish. Thus J_{m-1}/J_m is a quotient of I^{m-1}/J_m, hence of I^{m-1}/I^m, hence is annihilated by I, as claimed.

Now the fact that J_{m-1}/J_m is an $\mathcal{O}_C$-module and has $H^0(J_{m-1}/J_m) = 0$ implies that J_{m-1}/J_m is torsion-free, hence locally free, as $\mathcal{O}_C$-module. Moreover, $H^1(J_{m-1}/J_m) = 0$ as well, as follows from the exact sequence

$$0 \to J_{m-1}/J_m \to I/J_m \to I/J_{m-1} \to 0$$

and the fact that $H^1(I/J_m) = H^1(I) = 0$ and $H^0(I/J_{m-1}) = 0$. Finally, it follows from the classification of vector bundles on $\mathbf{P}^1$ that any such bundle with vanishing H^0 and H^1 must be a direct sum of $\mathcal{O}(-1)$'s, proving Claim 1.

Proof of Claim 2. We will prove by induction on m that

$$H^1(\Omega_{C_m} \otimes K) = 0.$$

Define ideals $L_m, I_m \subset \mathcal{O}_{C_m}$ by

$$\mathcal{O}_{C_{m-1}} = \mathcal{O}_{C_m}/L_m, \qquad \mathcal{O}_C = \mathcal{O}_{C_m}/I_m.$$

Thus by assumption

$$L_m \cdot I_m = 0, \qquad L_m = lm\mathcal{O}_C(-1)$$

(L_m may be called the *socle* of C_m).

Consider the natural exact sequence

$$0 \to Q \to \Omega_{C_m} \to \Omega_{C_{m-1}} \to 0. \tag{3.4}$$

By induction, it will suffice to show $H^1(Q \otimes K) = 0$. Note that Q has a natural subsheaf consisting of elements divisible by an element of L_m, with the quotient generated by symbols of the form $d\varepsilon$ for $\varepsilon \in L_m$. Note moreover that $\varepsilon d\varepsilon = \frac{1}{2}d(\varepsilon^2) = 0$ for $\varepsilon \in L_m$. We therefore have an exact sequence

$$0 \to A_{m-1} \otimes L_m \to Q \to L_m \to 0, \tag{3.5}$$

where $A_{m-1=\Omega_{C_{m-1}}} \otimes \mathcal{O}_C$ and $A_{m-1} \otimes L_m = L_m \cdot \Omega_{C_m}$. We have an exact sequence

$$0 \to B_{m-1} \to A_{m-1} \to \Omega_C \to 0.$$

By induction, we may assume B_{m-1} is a quotient of a sum of $\mathcal{O}(-1)$'s, hence $H^1(B_{m-1}) = 0$; then applying (3.4) and (3.5) yields the same for

B_m. Consider the following diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & B_{m-1}\otimes L_m & = & B_{m-1}\otimes L_m & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & A_{m-1}\otimes L_m & \to & Q & \to & L_m & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \Omega_C\otimes L_m & \to & \overline{Q} & \to & L_m & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{3.6}$$

Clearly $H^1(B_{m-1}\otimes L_m\otimes K)=0$, so it will suffice to prove that $H^1(\overline{Q}\otimes K)=0$, and this moreover is clear from (3.6) except in the one case $K=\mathscr{O}_C(1)$, in which it will suffice to prove that the coboundary map

$$\partial: H^0(L_m\otimes K)=H^0(l_m\mathscr{O}_C)\to H^1(\Omega_C\otimes L_m\otimes K)=H^1(l_m\Omega_C)$$

is an isomorphism. But this follows easily provided we can identify $\overline{Q}$ with the *principal parts sheaf* of L_m, i.e. it suffices to prove

$$\overline{Q}\simeq \mathscr{P}^1_C(L_m). \tag{3.7}$$

To prove (3.7), note first that $\overline{Q}$ is indeed an $\mathscr{O}_C$-module, i.e. is annihilated by I_m (although Q is not): This follows by observing (3.6) and using the identity

$$x d\varepsilon = -\varepsilon dx \in B_{m-1}\otimes L_m, \qquad \varepsilon\in L_m,\ x\in I_m.$$

Now given that $\overline{Q}$ is an $\mathscr{O}_C$-module which is an extension of L_m by $\Omega_C\otimes L_m$, we may identify it as $\mathscr{P}^1_C(L_m)$ if we can construct a differential operator of order exactly 1, $\nabla: L_m\to\overline{Q}$ lifting the identity on L_m. But ∇ may simply be defined as follows. Let $d:\mathscr{O}_{C_m}\to\Omega_{C_m}$ be the usual derivation and note that $d(L_m)\subseteq Q$. We may then define ∇ as the composite

$$L_m \xrightarrow{d} Q \to \overline{Q}, \qquad \nabla = (L_m \to \overline{Q}).$$

Remark 3.4. Given a good filtration $C_\bullet$ as above, we may define the *width* as $l=\sum l_m$; this is just the generic length of $\mathscr{O}_{\tilde{C}}$. Then the foregoing argument yields the estimate $h^1(\Omega_{\tilde{C}})\le l$. Thus if in the situation of Theorem 3.2 we consider canonically trivial, rather than canonically positive, contractions $f: X \to Y$, we may conclude that the codimension of $\mathrm{Def}(X, f, Y)$ in $\mathrm{Def}(Y)$ is at most l.

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